MOTION OF A FINITE MASS OF FLUID

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The qualitative properties of the nonsteady motion of a finite volume of fluid completely bounded by a free surface is discussed in this paper. The motion arises from a specified initial state. The external body forces are known functions of the coordinates and the time. The fluid can be viscous or ideal, possessing surface tension or devoid of it.

1. Statement of the Problem. The problem of the motion of a finite mass of fluid reduces to a search for a region $\Omega_t \in R^3$ and a solution $\mathbf{v}(\mathbf{x}, t) = (v_1(\mathbf{x}, t), v_2(\mathbf{x}, t), v_3(\mathbf{x}, t))$, and $p(\mathbf{x}, t)$ of the Navier–Stokes system of equations

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla v = -\nabla p + \nu\Delta\mathbf{v} + \mathbf{g}(\mathbf{x}, t),\ \operatorname{div}\mathbf{v} = 0 \qquad (1.1)$$

in this region so that on the boundary Γ_t of the region Ω_t the boundary conditions

$$\mathbf{v}\cdot\mathbf{n}_{\Gamma_t} = V_n \ \text{ for } \ \mathbf{x} \in \Gamma_t; \qquad (1.2)$$

$$p\mathbf{n}_{\Gamma_t} - 2\nu D\mathbf{n}_{\Gamma_t} = 2\sigma H\mathbf{n}_{\Gamma_t} \ \text{ for } \ \mathbf{x} \in \Gamma_t, \qquad (1.3)$$

and the initial condition at t = 0

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}),\ \mathbf{x} \in \Omega_0 \equiv \Omega \qquad (1.4)$$

are satisfied. The region Ω is assumed to be specified and bounded.

Here v is the velocity vector, p is the fluid pressure, g is the acceleration of the external body forces, $\nu \geq 0$ is the viscosity of the fluid, and $\sigma \geq 0$ is the surface tension coefficient. The quantities ν and σ are assumed to be constants, and the vector g is assumed to be a known function of x and t. The fluid density is assumed to be equal to unity. The displacement velocity of the surface Γ_t in the direction of the outer normal is denoted in (1.2) by V_n, and the unit vector of the outer normal to Γ_t is denoted by $\mathbf{n}_{\Gamma_t}$. If the surface Γ_t is specified by the equation $F(\mathbf{x}, t) = 0$, then $V_n = -F_t/|\nabla F|$. In the condition (1.3) D is the strain rate tensor with elements $D_{ik} = (\partial v_i/\partial x_k + \partial v_k/\partial x_i)/2$ (i, k = 1, 2, 3), and H is double the mean curvature of the surface Γ_t. It is assumed that $H > 0$ if Γ_t is convex external to the fluid.

Eq. (1.2) indicates that the surface Γ_t bounds the fluid volume Ω_t. According to (1.3), external surface forces on the boundary of the region Ω_t are absent, i.e., the boundary Γ_t is a free surface.

The vector field $\mathbf{v}_0$ in (1.4) is assumed to be specified and solenoidal:

$$\operatorname{div}\mathbf{v}_0 = 0,\ \mathbf{x} \in \Omega. \qquad (1.5)$$

If $\nu > 0$, then $\mathbf{v}_0$ still satisfies the congruence condition with (1.3):

$$D\mathbf{n}_{\Gamma_t} - (\mathbf{n}_{\Gamma_t}\cdot D\mathbf{n}_{\Gamma_t})\,\mathbf{n}_{\Gamma_t} = 0,\ \mathbf{x} \in \Gamma_0,\ t = 0. \qquad (1.6)$$

The main difficulty with investigating the problem (1.1)-(1.4) consists of the necessity of seeking the region Ω_t. However, the characteristics of the condition (1.2) permit transforming this problem into another one in which the region in which the solution is defined is specified in advance. This situation is achieved by conversion to Lagrangian coordinates.

The trajectory of a particle located at time t = 0 at the point ξ is specified by the formula

$$\mathbf{x} = \mathbf{x}(\xi, t), \qquad (1.7)$$

in which the functions $x_i(\xi, t)$ (i = 1, 2, 3) are determined from the Cauchy problem

$$d\mathbf{x}/dt = \mathbf{v}(\mathbf{x}, t),\ \mathbf{x} = \xi \ \text{ at } \ t = 0. \qquad (1.8)$$

The variables $\xi = (\xi_1, \xi_2, \xi_3)$ are called Lagrangian. If the relationship $F(\mathbf{x}(\xi, t), t) \equiv f(\xi, t) = 0$ defines the free boundary, then it follows from (1.2) and (1.8) that $f_t = 0$ and the equation of a free boundary in Lagrangian coordinates is simply $f(\xi) = 0$. Therefore, if one constructs Γ_t as the shape $\Gamma_0 \equiv \Gamma$ with the mapping (1.7), condition (1.2) will be automatically satisfied.

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Let us formulate the problem (1.1)-(1.4) in Lagrangian coordinates in the case $\nu=0$ and $\sigma=0$ (an ideal fluid with zero surface tension). It is required to find the vector $\mathbf{x}(\xi, t)$ and the function $p(\xi, t)$ in the region $\Omega\times[0, T]$ so that the following equations, the initial and boundary conditions, are satisfied:

$$M^{*}\mathbf{x}_{tt}+\nabla_{\xi}p=\mathbf{g}(\mathbf{x}, t),\ \det M=1; \tag{1.9}$$

$$p=0 \text{ for } \xi\in\Gamma; \tag{1.10}$$

$$\mathbf{x}=\xi,\ \mathbf{x}_t=\mathbf{v}_0(\xi) \text{ at } t=0, \tag{1.11}$$

and $\operatorname{div}_\xi \mathbf{v}_0=0$. Here ∇_ξ and div_ξ are the gradient and the divergence with respect to the variables (ξ_1, ξ_2, ξ_3), M is the Jacobian matrix of the mapping (1.7) for t fixed, and $M_{ik}=\partial x_i/\partial\xi_k$ (i, k=1, 2, 3).

This article is a review of papers which have been devoted to the investigation of the problem (1.1)-(1.4) in the exact formulation, as well as of some models of it. The results of investigations conducted during the last 15 years through the initiative and under the direction of L. V. Ovsyannikov comprise the basis of this article.

2. Existence Theorems. The first (and still the only) result on the solvability of the problem (1.9)-(1.11) belongs to L. V. Ovsyannikov [1] and is based on the theory which he developed of the nonlinear Cauchy problem on the scale of Banach spaces [2].

Let Ω be a simply connected two-dimensional region, $\mathbf{v}_0$ be an irrotational two-dimensional vector field in Ω, and $g(x, t)=0$. In this case the problem (1.9)-(1.11) is equivalent to the following:

$$\text{for } |\zeta|=1,\ t>0\quad h_t=\zeta h_\zeta\frac{1}{2\pi i}\int\limits_{|\tau|=1}\frac{\tau+\zeta}{\tau-\zeta}\left(\operatorname{Re}\frac{\tau w_\tau}{|h_\tau|^2}\right)\frac{d\tau}{\tau}, \tag{2.1}$$

$$w_t=\zeta w_\zeta\frac{1}{2\pi i}\int\limits_{|\tau|=1}\frac{\tau+\zeta}{\tau-\zeta}\left(\operatorname{Re}\frac{\tau w_\tau}{|h_\tau|^2}\right)\frac{d\tau}{\tau}-\frac{1}{4\pi i}\int\limits_{|\tau|=1}\frac{\tau+\zeta}{\tau-\zeta}\left|\frac{w_\tau}{h_\tau}\right|^2\frac{d\tau}{\tau};$$

$$\text{for } |\zeta|\leqslant 1,\ t=0\ w=w_0(\zeta),\ h=h_0(\zeta), \tag{2.2}$$

where $h(\zeta, t)$ is the conformal mapping of the unit circle of the complex plane ζ onto the region Ω_t of the plane $z=x_1+ix_2$, and $w(\zeta, t)\equiv w^*(z, t)$ is the complex potential of the motion.

We define the Banach space $B_\rho(\rho>0)$ as the set of traces on the circle $|\zeta|=1$ of the analytic functions $u(\zeta)$ for which the norm

$$\|u\|_\rho=\sum_{n=-\infty}^{\infty}|u_n|\,e^{|n|\rho}$$

is finite, where u_n are the coefficients of the Laurent series of the function $u(\zeta)$, which is analytic in the ring $e^{-\rho}<|\zeta|<e^{\rho}$. The set $S=\bigcup_{0<\rho}B_\rho$ is the scale of the Banach spaces. The following theorem [1] is valid:

If the initial data (2.2) are such that

$$w_0(\zeta),\ h_0(\zeta),\ 1/h_0'(\zeta)\in B_{\rho_0},$$

then a constant $K>0$ is found such that the solution of the problem (2.1) and (2.2) exists and belongs to the space B_ρ for any $\rho<\rho_0$ for values of t satisfying the inequality

$$\rho+K|t|<\rho_0. \tag{2.3}$$

This solution is unique in the scale S and is holomorphic with respect to t at the point $t=0$ in the region (2.3).

There are still no other results of any kind on the solvability of the general problem (1.9)-(1.11). We note in this connection that an existence and uniqueness theorem of a solution of the three-dimensional Cauchy–Poisson problem similar to (1.9)-(1.11) on water waves in classes of analytic functions has been proven in [3]. A proof of the solvability of the problem (1.9)-(1.11) in a class of functions of finite smoothness encounters significant difficulties, whose nature is explained below (also see [4]).

The problem (1.1)-(1.4) with $\nu>0$ and $\sigma=0$ has been investigated in [5, 6], where the condition (1.3) with $\sigma=0$ is replaced by a more general one, $p\mathbf{n}-2\nu D\mathbf{n}=p_0(\mathbf{x}, t)\mathbf{n}$ ($p_0(\mathbf{x}, t)$ is a specified distribution of the external pressure on the fluid surface). Lacking the possibility here of giving a complete presentation of the results of [5, 6], we restrict ourselves to a reduced formulation of one of them.

Let $\Gamma\in C^{2+\alpha}$, $\mathbf{v}_0\in C^{2+\alpha}(\Omega)$ with some $\alpha\in(0, 1)$, $g=0$, and $p_0=0$, and let the congruence conditions (1.5) and (1.6) be satisfied. We will denote the Holder norm $|\mathbf{v}_0|_\Omega^{(2+\alpha)}=R$. For any $R>0$ is found a $T>0$ such that the problem (1.1)-(1.4) with $\nu > 0$ and $\sigma = 0$ has a unique solution for $t \in [0, T]$, and **v** and p belong to certain Holder classes. In addition, one can find for any T > 0 an R > 0 such that the problem (1.1)-(1.4) is uniquely soluble on the interval [0, T].

We note that all the theorems mentioned above on the solvability of the problems of the nonsteady motion of a fluid with a free boundary are of a local nature in the exact formulation. This restriction is associated with the essence of the matter. One can describe a situation in which in the process of motion two points of the free surface which are initially located at a finite distance approach one another with subsequent impact of one part of the fluid with the other. The mathematical nature of the peculiarities which arise is complicated and has not been investigated up to this time. Also the problem of the possible loss of smoothness of the free surface as time progresses has not been investigated. Finally, not a single exact result of a general nature exists concerning the solvability of the problem (1.1)-(1.4) in the case $\sigma \neq 0$.

3. Finite-Dimensional Models. Below are enumerated examples of solutions of the problem (1.1)-(1.4) for which the search reduces to the integration of a system of ordinary equations. The richest class of such solutions is allowed in the case $\nu = 0$ and $\sigma = 0$. These involve motions with a linear velocity field discovered by Dirichlet (see [7]) and investigated in detail in [8] (also see [9, 10]). The mapping (1.7) is given here by the formula

$$\mathbf{x} = A(t)\boldsymbol{\xi} + \mathbf{x}_0(t), \tag{3.1}$$

so that M = A. It is necessary for the existence of such solutions that the vector of the external forces be a linear function of **x**:

$$\mathbf{g} = G(t)\mathbf{x} + \mathbf{g}_0(t). \tag{3.2}$$

By virtue of (1.9)-(1.11) the matrix A and the vector $\mathbf{x}_0$ satisfy the equations

$$A'' - GA = qA^{*-1}N; \tag{3.3}$$

$$\mathbf{x}_0'' - G\mathbf{x}_0 = \mathbf{g}_0 + qA^{*-1}\mathbf{b} \tag{3.4}$$

and the initial conditions

$$A(0) = E, \quad A'(0) = A_0'; \tag{3.5}$$

$$\mathbf{x}_0(0) = 0, \quad \mathbf{x}_0'(0) = \mathbf{x}_{00}'. \tag{3.6}$$

Here A_0' is an arbitrary constant matrix with $\mathrm{Sp}A_0' = 0$, $N = \mathrm{diag}\{n_1, n_2, n_3\}$, where n_1, n_2, n_3 are arbitrary positive numbers, b and $\mathbf{x}_{00}'$ are arbitrary constant vectors, and the notation

$$q = \frac{\mathrm{Sp}\,(A^{-1}A')^2 - \mathrm{Sp}\,(A^{-1}GA)}{\mathrm{Sp}\,(A^{-1}A^{*-1}N)}$$

is introduced. The formula for the pressure is of the form

$$p = q(l + 2\mathbf{b}\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot N\boldsymbol{\xi})/2,$$

where l is a constant. The free surface Γ is an ellipsoid with the equation

$$l + 2\mathbf{b}\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot N\boldsymbol{\xi} = 0.$$

We will restrict ourselves in the following to motions in which g = 0, b = 0, and $\mathbf{x}_{00}' = 0$. By virtue of (3.2), (3.4), and (3.6), $\mathbf{x}_0(t) = 0$.

Cauchy's problem (3.3) and (3.5) is uniquely solvable for all t [9]. The system (3.3) with G = 0 has eight integrals which express conservation of the mass (det A = 1), energy, circulation, and angular momentum of a deformed fluid ellipsoid [7]. There exist a number of exact solutions of the problem (3.1). We will discuss one of them, which is found in [8].

Let N = E, and let the matrix A_0' have nonzero elements $2(a_0')_{22} = 2(a_0')_{33} = -(a_0')_{11} = -2b$, and $(a_0')_{32} = -(a_0')_{23} = \omega$. Then the matrix A has nonzero elements $a_{11} = m$, $a_{22} = a_{33} = k$, and $a_{32} = -a_{23} = n$, and

$$k = \frac{1}{\sqrt{m}}\cos\left(\omega\int_0^t m(\tau)\,d\tau\right), \quad n = \frac{1}{\sqrt{m}}\sin\left(\omega\int_0^t m(\tau)\,d\tau\right),$$

and the function m(t) is found by quadrature from the equation

$$m'^2 = 4m^3\,\frac{3b^2 + \omega^2(1-m)}{1+2m^3} \tag{3.7}$$

with the condition m(0) = 1. The equation of the free boundary Γ_t in Eulerian coordinates is

$$\frac{x_1^2}{m^2} + m(x_2^2 + x_3^2) = c^2 \equiv l.$$

The interpretation of the solution is as follows. At the initial instant the fluid filled a sphere Γ and was in a state of uniform rotation on which an irrotational linear velocity field was superimposed. Let us assume for definiteness that $\omega \neq 0$ and $b > 0$. Then during $0 < t < t_*$ the sphere Γ is elongated into an ellipsoid of revolution Γ_t with axis x_1 until its semimajor axis takes on the maximum value $cm_* = c[3(b/\omega)^2 + 1]^{1/2}$ at the instant $t = t_*$. After this the ellipsoid starts to contract, passes the shape of the sphere Γ at the time $t = 2t_*$, and then shrinks to the plane $x_1 = 0$, merging with it as $t \to \infty$.

Now let us consider the case $\omega = 0$. The matrix A is diagonal, and the motion is irrotational. If $b > 0$, then $m > 1$ for $t > 0$ and $m \to \infty$ as $t \to \infty$. Thus when $b > 0$ and $\omega = 0$, the ellipsoid Γ_t expands in the direction of the x_1-axis, shrinking to this axis when $t \to \infty$. From the point of view of the stability of the motion this result indicates that the irrotational motion specified by the matrix A for $\omega = 0$ is unstable with respect to as small vortical perturbations as desired.

The question of the behavior of the solutions of the Cauchy problem (3.3) and (3.5) as $t \to \infty$ has not yet been solved. The hypothesis that when $A_0' \neq 0$ this problem has no bounded solutions seems plausible.

Now let us discuss two-dimensional motions with a linear velocity field. In this case ξ and x denote two-dimensional vectors, and A, A_0', and N are second-order matrices. The solution of the problem (3.3) and (3.5) here describes the motion of a rotating deformed ellipse. Using the motion integrals, it is possible to integrate this problem in quadratures [11]. It turns out that if the initial state is not rest, the following alternative occurs. Either one of the semiaxes of the ellipse increases indefinitely as $t \to \infty$ or the motion is uniform rotation of a fluid circle about its center.

We will consider the two-dimensional problem (3.3) and (3.5) with

$$A_0' = \begin{pmatrix} b & \omega \\ -\omega & -b \end{pmatrix}$$

and N=E (Γ is a circle of radius c). In this case the semiaxes of the ellipse $a_1(t)$ and $a_2(t)$ are related by the equations

$$\frac{1}{2}\left(a_1'^2 + a_2'^2\right) + \frac{4c^4\omega^2}{(a_1 + a_2)^2} = (b^2 + \omega^2)\,c^2, \qquad (3.8)$$
$$a_1 a_2 = c^2,\ a_1(0) = a_2(0) = c,$$

and the angular rotational velocity of the ellipse is equal to $4c^2\omega(a_1 + a_2)^{-2}$. The case $b = 0$ corresponds to rotation of the circle as a rigid body. It is evident from (3.8) that an initial deformation of the velocity field as small as desired ($b \neq 0$) disrupts the indicated steady motion.

The supply of exact solutions of the problem (1.1)-(1.4) in the case $\nu \neq 0$ and $\sigma \neq 0$ is extremely sparse. The only nontrivial example is the solution which describes the radial motion by inertia of a spherical layer [12-15]. The two-dimensional analogue of this solution describes the radial motion of a circular ring [13, 16]. The two-dimensional problem of rotationally symmetric motion of a rotating ring is more general.

4. Rotating Ring. We will discuss the two-dimensional problem (1.1)-(1.4) with special initial data: Ω is the circle $r_{10} < r = |x| < r_{20}$,

$$v_r = \chi_0 r^{-1},\ v_\theta = v_0(r) \quad \text{at} \quad t = 0,\ r_{10} < r < r_{20}, \qquad (4.1)$$

where v_r is the radial and v_θ is the circumferential component of the velocity in the polar coordinate system (r, θ), χ_0 is a specified constant, and v_0 is a specified function. The solution of this problem has the form

$$v_r = \chi(t) r^{-1},\ v_\theta = v_\theta(r, t),\ p = p(r, t),$$

and the equations of the free boundaries are: $r = r_i(t)$, $i = 1, 2$. This solution is interpreted as the motion of a fluid rotating ring under the action of inertial, viscous, and surface tension forces.

The problem (1.1)-(1.3) and (4.1) for $\nu > 0$ reduces to the solution of the associated system of one parabolic and three ordinary equations for the functions v_θ, χ, and r_i and to quadrature for finding p. This problem was investigated in [17], where the case $\sigma = 0$ is discussed, and in [18]. The results of these papers are detailed below.

Let us denote the angular momentum of the ring by L and its area by Σ. The quantities Σ and L are integrals of the motion. We introduce the dimensionless parameter

$$\beta = L^2/\rho\sigma\Sigma^{5/2}$$

(ρ is the fluid density). First let us assume that $\sigma > 0$. Then when the inequality $\beta > \beta_* \approx 5.17$ is satisfied, the problem has two steady solutions describing motion of the ring as a rigid body. If $\beta < \beta_*$, no steady solutions exist.

Let us assume that $v_0 \in C^{2+\alpha}[r_{10}, r_{20}]$ and the congruence conditions $v_0'(r_{10}) = v_0'(r_{20}) = 0$ are satisfied. If $\beta < \beta_*$, a value t_0 (finite or infinite) is found so that $r_1(t) \to 0$ as $t \to t_0$.

If $\beta \geq \beta_*$, two modes of motion are possible: vanishing of the inner radius and stabilization of the motion to rotation of the ring as a rigid body. The sufficient conditions for realization of each of these modes are given in [18]. For example, with the condition

$$8r_{20}^2 \sum \rho E_0 < L^2,$$

where E_0 is the total energy of the fluid at the time $t=0$, the inner radius of the ring vanishes.

In the case $\sigma = 0$ the qualitative picture of the motion changes significantly. If $L \neq 0$ and $\chi_0 < 0$, in the condition (4.1), the rotating ring first converges to the center until the inner radius reaches a positive minimum. Then divergence of the ring begins at once. There are two different divergence modes. If both inequalities

$$\frac{\chi_0}{\nu} < 4, \quad \frac{r_{10}^2}{\sum v^2} \int_{r_{10}}^{r_{20}} r v_0^2(r)\, dr < 2, \tag{4.2}$$

are satisfied, then $r_1 = O(\sqrt{t})$ as $t \to \infty$. If even one of these inequalities is replaced by the opposite one, then $r_1(t) = O(t)$ as $t \to \infty$ [17].

The case $v_0 = 0$ in the condition (4.1) corresponds to purely radial motion of the ring. If at the same time $\sigma \neq 0$, the ring either diverges to infinity or its inner radius vanishes at some instant. If $\sigma = 0$ and the first of the inequalities (4.2) is satisfied, $\lim r_1(t) = r_{1\infty} > 0$ exists as $t \to \infty$. In the opposite case $r_1 \to \infty$ as $t \to \infty$.

Thus far we have been dealing with the motion of a viscous ring. In the case $\nu = 0$ the problem simplifies and permits integration in quadratures [9]. Here five qualitatively different modes of motion occur, depending on the initial data. In particular, when $\sigma \neq 0$ and $L \neq 0$, radial self-oscillations of a rotating ring of ideal fluid are possible.

5. Steady Motions. Steady motions of an isolated volume of viscous capillary fluid permit a simple description: The fluid rotates as a rigid body about an axis parallel to a specified angular momentum vector, and the free surface is motionless in a rotating coordinate system. It is determined as a closed minimal surface in the centrifugal force field which restricts the specified volume.

The problems of the existence, stability, and branching of the equilibrium shapes of a rotating fluid are detailed at great length in [19] and will not be discussed in this article. If the fluid lacks surface tension, then steady motion of an isolated volume can only be translational when $\nu \neq 0$. In the two-dimensional case rotation of a circle and a ring as a rigid body is also allowed. If $\sigma = 0$ and $\nu = 0$ simultaneously, nontrivial steady motions of an isolated fluid volume are possible. An example of such motion [20] is constructed below. In this example the flow of an ideal fluid is rotationally symmetric with the z-axis, and its vorticity is proportional to the distance r to this axis.

If one denotes the stream function by ψ and the radial and axial velocities by v_r and v_z, then $v_r = -r^{-1}\partial\psi/\partial z$, $v_z = r^{-1}\partial\psi/\partial r$, and ψ satisfies the equation

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) = kr, \tag{5.1}$$

where k is a constant. The circumferential velocity v_θ is equal to cq/r, where cq is a constant. For flows of the indicated kind Euler's equations allow the integral

$$p + \frac{1}{2}\left(v_r^2 + v_z^2 + c^2q^2/r^2\right) + k\psi = \frac{1}{2}c^2 \equiv \text{const}. \tag{5.2}$$

We will look for free surfaces Γ similar to a torus. Let us denote the meridional cross section of Γ by γ and the plane region bounded by the curve γ by ω. The conditions on the free boundary $p = 0$, $\psi = 0$, and Eq. (5.2) lead to the equations

$$\psi|_\gamma = 0, \quad \frac{1}{r}\frac{\partial \psi}{\partial n}\Big|_\gamma = c\sqrt{1 - q^2/r^2}, \tag{5.3}$$

where $\partial/\partial n$ is differentiation with respect to the direction of the outer normal to γ. By virtue of (5.1) and (5.3), the constants k and c are related by the equation

$$k = c\int\limits_{\gamma} (1 - q^2/r^2)^{1/2} d\gamma \Big/ \int\limits_{\omega} r d\omega.$$

Let us switch in (5.1) and (5.3) to the new variables

$$r = a + bx, \ z = by, \ \psi(r, z) = abc(1 - \mu^2)^{1/2} w(x, y)$$

(a and b are some constants with the dimension of length, and $\mu = q/a$), and let us set $\varepsilon = b/a$. The shapes of the curve γ and the region ω on the x, y plane are denoted by γ_0 and ω_0, respectively. In the new variables the problem (5.1) and (5.3) takes the form

$$\Delta w - \frac{\varepsilon}{1 + \varepsilon x} \frac{\partial w}{\partial x} = \frac{(1 + \varepsilon x)^2}{(1 - \mu^2)^{1/2}} \frac{\int\limits_{\gamma_0} [1 - \mu^2 (1 + \varepsilon x)^{-2}]^{1/2} d\gamma_0}{\int\limits_{\omega_0} (1 + \varepsilon x) \, d\omega_0}, \tag{5.4}$$

$$w|_{\gamma_0} = 0, \ \left. \frac{\partial w}{\partial n_0} \right|_{\gamma_0} = \frac{(1 + \varepsilon x)}{(1 - \mu^2)^{1/2}} \left[1 - \mu^2 (1 + \varepsilon x)^{-2}\right]^{1/2}$$

(Δ is the Laplacian in the variables x and y).

If $\varepsilon = 0$, the problem (5.4) has a one-parameter family of solutions in which γ_0 is the circle $x^2 + y^2 = c^2$ and $w = (x^2 + y^2 - c^2)/2c$. It has been demonstrated that for sufficiently small ε in the case of some dependence $\mu = \mu(\varepsilon)$ this problem has a three-parameter family of solutions. A four-parameter family of solutions of the problem (5.1) and (5.3) corresponds to it. The kinetic energy of the fluid, the moment of inertia of the meridional cross section of the free surface with respect to the straight line r = const which passes through the center of gravity of the cross section, the length of the meridional cross section of the free surface, and the distance from the center of gravity of the cross section to the symmetry axis (the ratio of the last two numbers should be sufficiently small) can be selected as the determining physical parameters of this family.

6. Small Perturbations. First, we will discuss the motion of an ideal fluid with zero surface tension. Such motion is determined by the solution $x(\xi, t)$ and $p(\xi, t)$ of the problem (1.9)-(1.11) in some cylinder $\overline{Q}_T = \overline{\Omega} \times [0, T]$. The solution x and p corresponding to the initial velocity field $x_t(\xi, 0) = v_0(\xi)$, div $v_0 = 0$, will be called the fundamental one.

Let us consider in this cylinder Q_T another solution $\tilde{x}$ and $\tilde{p}$ of the problem (1.9)-(1.11) with the initial velocity field

$$\tilde{v}_0(\xi) = v_0(\xi) + V_0(\xi), \ \operatorname{div} V_0(\xi) = 0.$$

The solution $\tilde{x}$ and $\tilde{p}$ is called the perturbed solution, and the function V_0 is called the initial perturbation. Let us set $\tilde{x} = x + X$, $\tilde{p} = p + \nabla_\xi p \cdot M^{-1} X + P$, and let us call the functions X and P the perturbations of the fundamental solution. Assuming smallness of the initial perturbation, one can hope that the functions X and P will be small for some time interval. Substituting the expressions for $\tilde{x}$ and $\tilde{p}$ into Eqs. (1.9)-(1.11) and discarding terms which are nonlinear in the perturbations, we arrive at a linear problem for the functions X and P.

The linear model in the theory of nonsteady motions of a fluid with a free boundary is of interest for two reasons. In the first place, linearization on the solution of the problem with a free boundary offers the possibility of understanding the mathematical nature of this problem. In the second place, if there exists some solution which is defined for all $t > 0$, then analysis of the behavior of small perturbations as $t \to \infty$ will permit assessing the stability of this solution.

An enormous number of papers has been devoted to the investigation of small perturbations of the rest or uniform rotation of a fluid. Particularly relevant here are papers on the linear wave theory, as well as classical papers on the determination of the figure of the Earth, which go back as far as the time of Newton. However, there has been an absence up until recently of papers in which the problem of small perturbations of an arbitrary solution of Euler's equations in a region with a partially or completely free boundary has been studied. The formulations of this problem and the first results of its investigation are given in [13], where the case of the irrotational motion of a fluid is discussed. The general problem of small perturbations of the motion of an ideal fluid with a free boundary in an irrotational field of body forces has been investigated in [21]. This problem reduces in the case of a free boundary to a search for a single function $\Phi(\xi, t)$ which satisfied the following equations:

$$\operatorname{div} [M^{-1} M^{*-1} (\nabla\Phi + V_0)] = - \operatorname{div}\left[(M^{-1} W)_t \int\limits_0^t W^{-1} M^{*-1} \times (\nabla\Phi + V_0) \, dt \right] \quad (\xi \in \Omega, \ 0 \leqslant t \leqslant T); \tag{6.1}$$

$$(a\Phi_t)_t + \mathbf{n}\cdot M^{-1}M^{*-1}(\nabla\Phi + \mathbf{V}_0) = -\mathbf{n}\cdot(M^{-1}W)_t \int_0^t W^{-1}M^{*-1} \times (\nabla\Phi + \mathbf{V}_0)\,dt \quad (\xi \in \Gamma,\ 0 < t < T); \tag{6.2}$$

$$\Phi = 0,\ \Phi_t|_{t=0} = 0 \quad (\xi \in \Omega,\ t = 0), \tag{6.3}$$

where $a = -(\partial p/\partial n)^{-1}$, $\mathbf{n}$ is the unit vector of the external normal to the boundary Γ of the region Ω, and $\partial p/\partial n$ is the derivative of the pressure p with respect to the normal $\mathbf{n}$ to Γ. The matrix $W(\xi, t)$ is the solution of the problem

$$W_t = \left(\frac{\partial(\mathbf{v})}{\partial(\mathbf{x})}\right)^* W,\quad W|_{t=0} = E, \tag{6.4}$$

$\mathbf{v} = \mathbf{x}_t$, and M is the matrix of the Jacobian of the mapping (1.7). The operator ∇ at this point denotes the gradient with respect to the Lagrangian variables ξ_1, ξ_2, and ξ_3. If the function $\Phi(\xi, t)$ is known, then the perturbation of the pressure P is determined as $P = -\Phi_t$, and the vector $\mathbf{X}$ is given by the integral

$$\mathbf{X} = W\int_0^t W^{-1}M^{*-1}(\nabla\Phi + \mathbf{V}_0)\,dt.$$

We note that the function $\Phi(\xi, t)$ should satisfy the condition

$$\int_\Gamma \left(\frac{\partial p}{\partial n}\right)^{-1} \Phi_t\,d\Gamma = 0,$$

which follows from (6.1) and (6.2) when $\partial p/\partial n \neq 0$.

The existence and uniqueness theorem of the generalized problem is valid for the problem (6.1)-(6.3). We will restrict ourselves here to the case in which the fundamental and perturbed motion are irrotational. At the same time $\mathbf{V}_0 = \nabla\Phi_0$, where $\Delta\Phi_0 = 0$, the matrices M and W coincide, and the following problem is obtained for the function $\Psi = \Phi + \Phi_0$:

$$\operatorname{div}(M^{-1}M^{*-1}\nabla\Psi) = 0 \quad \text{for} \quad \xi \in \Omega,\ 0 \leqslant t \leqslant T; \tag{6.5}$$

$$(a\Psi_t)_t + \mathbf{n}\cdot M^{-1}M^{*-1}\nabla\Psi = 0 \quad \text{for} \quad \xi \in \Gamma,\ 0 < t < T; \tag{6.6}$$

$$\Psi = \Phi_0(\xi),\ \Psi_t = 0 \quad \text{for} \quad t = 0,\ \xi \in \overline{\Omega}. \tag{6.7}$$

This problem can, in its turn, be reduced to the Cauchy problem for a differential equation with an unbounded nonlocal operator in the Sobolev space $W_2^{1/2}(\Gamma)$

$$(a\psi_t)_t + K(t)\psi = 0; \tag{6.8}$$

$$\psi = \psi_0,\ \psi_t = 0 \quad \text{at} \quad t = 0 \tag{6.9}$$

with the desired function $\psi = \Psi|_\Gamma$. The operator K compares the functions $\psi \in W_2^{1/2}(\Gamma)$ and the element $K\psi \in W_2^{-1/2}(\Gamma)$ according to the rule: With respect to the function ψ a solution of Eq. (6.5) is sought with the condition $\Psi|_\Gamma = \psi$, and then $K\psi = \mathbf{n}\cdot M^{-1}M^{*-1}\nabla\Psi|_\Gamma$ is calculated. The notation K(t) emphasizes the dependence of the operator K on t (this is associated with the fact that M depends on t). In the initial condition (6.9) ψ_0 is the trace of the function $\Phi_0(\xi) \in W_2^1(\Omega)$ on the surface Γ.

One can show that the operator K is symmetrical and positive definite on the subspace of the Hilbert space $W_2^{1/2}(\Gamma)$ formed by the functions ψ with zero average value on Γ. We will assume that the condition

$$\left(-\frac{\partial p}{\partial n}\right)^{-1} \equiv a(\xi, t) \geqslant a_0 > 0 \tag{6.10}$$

is satisfied for all $\xi \in \Gamma$, and $t \in [0, T]$. Inthis case one can treat Eq. (6.8) as a hyperbolic pseudodifferential equation on the free boundary Γ.

If the fundamental solution is such that $\Gamma \in C^2$, $\mathbf{x}, p \in C^3(\overline{Q}_T)$ and (6.10) is satisfied, then the solution of (6.5)-(6.7) permits the a priori estimate [13, 22]

$$\int_\Gamma \Psi_t^2\,d\Gamma + \int_\Omega |\nabla\Psi|^2\,d\Omega \leqslant C(T)\int_\Omega |\nabla\Phi_0|^2\,d\Omega. \tag{6.11}$$

The solvability of the problem (6.5)-(6.7) under the condition (6.10) has been demonstrated in [22] with the use of this estimate. Under these conditions an existence and uniqueness theorem of a solution of the more general problem (6.1)-(6.3) has been established [22]. We note that the problem (6.1)-(6.3) can also be reduced to an operator equation of the type (6.8). This equation will now be inhomogeneous, and the operator K(t) will be nonlocal with respect to t and asymmetrical. However, its main part is a symmetrical and positive definite operator, so that the equation mentioned preserves the properties of the hyperbolic equation under the condition (6.10).

We emphasize that precisely the condition $\partial p/\partial n < 0$ for $\sigma = 0$ guarantees the correctness of the problem of small perturbations. The importance of this condition has been noted in [23, 13, 24]; the solvability of the two-dimensional Cauchy–Poisson problem in an exact formulation in the class of functions with finite smoothness has been demonstrated in [24], and a remarkable peculiarity of this problem has been established. It turns out that the Cauchy problem for linearized equations is correctly posed only when the linearization is carried out on the exact solution of the nonlinear equations. Evidently, this same peculiarity is characteristic of the problem of the motion of an isolated volume of ideal fluid devoid of surface tension.

Let us return to the problem of small perturbations (6.1)-(6.3). It is rather difficult to characterize the class of fundamental motions in which the inequality (6.10), which guarantees the correctness of the problem (6.1)-(6.3), is satisfied. We note, however, that for irrotational motions which are different from a constant flow in a solenoidal field of body forces this inequality is certainly satisfied [13, 9]. Physically, condition (6.10) indicates that the acceleration of the particles on the free boundary is directed inside the fluid.

For fundamental motions in which the inequality

$$\partial p/\partial n > 0 \tag{6.12}$$

occurs for all $\xi \in \Gamma$ and $t \in [0, T]$, Eq. (6.8) is "elliptical," and the Cauchy problem for it is formulated incorrectly in the Adamar sense. Condition (6.12) can be satisfied even for irrotational motions if the entire boundary of the region is not free or if external forces are acting on the fluid. Examples of the incorrectness of the problem of small perturbations with a plane free boundary are constructed in [23]; this fact was first noted already by Rayleigh (see, e.g., [25]). Concerning the motion of a fluid volume by inertia, strong vorticity of the motion may be the cause here of the incorrectness of the problem of small perturbations. Thus if one takes the solution from Sect. 3 as the fundamental one describing the motion of a rotating ellipsoid, then the inequality $\partial p/\partial n > 0$ is surely satisfied for t close to t_*.

When $\partial p/\partial n > 0$, one can hope for solvability of the linearized problem (6.8) or the more general problem only in the class of analytic functions; therefore, one should not expect solvability of the original nonlinear problem (1.9)-(1.11) in the case of arbitrary initial data in classes of functions of finite smoothness.

It is interesting to note that in the case in which the fundamental motion is uniform rotation of the fluid as a rigid body, the problem of small perturbations of it has been correctly formulated [26], although the inequality $\partial p/\partial n > 0$ is satisfied. This case is an exceptional one, since in a rotating coordinate system the fundamental motion is rest.

Surface tension has not been taken into account in the preceding discussions of this section. As has been noted [23], surface tension has proven to be the factor which stabilizes short-wavelength perturbations and makes the mathematical problem of small perturbations a correctly posed one.

The problem of small perturbations of an arbitrary motion of an ideal fluid which possesses surface tension has been discussed in [27]. The equations of small perturbations in Lagrangian coordinates have the form

$$\operatorname{div} M^{-1}\mathbf{V} = 0; \tag{6.13}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{\partial(\mathbf{v})}{\partial(\mathbf{x})}\mathbf{V} + \left\{\left[\frac{\partial(\mathbf{v})}{\partial(\mathbf{x})} - \left(\frac{\partial(\mathbf{v})}{\partial(\mathbf{x})}\right)^*\right]_t + \left(\frac{\partial(\mathbf{v})}{\partial(\mathbf{x})}\right)^2 - \left(\frac{\partial(\mathbf{v})}{\partial(\mathbf{x})}\right)^{*2} + M^{*-1}\left(\frac{\partial(\mathbf{g})}{\partial(\mathbf{x})}\right)^* - \frac{\partial(\mathbf{g})}{\partial(\mathbf{x})}\right\} M \int_0^t M^{-1}\mathbf{V}\,dt + M^{*-1}\nabla P = 0, \ \ \xi \in \Omega, \ \ t \geqslant 0; \tag{6.14}$$

$$P + \left(\frac{\partial p}{\partial n}\Big|_{\Gamma_t} + \sigma q^2\right) R + \sigma \bar{\Delta}_{\Gamma} R = 0; \tag{6.15}$$

$$R = \rho \int_0^t \mathbf{n} \cdot M^{-1}\mathbf{V}\,dt, \ \ \xi \in \Gamma, \ \ t > 0; \tag{6.16}$$

$$\mathbf{V}|_{t=0} = \mathbf{V}_0(\xi), \ \ \operatorname{div}\mathbf{V}_0 = 0, \tag{6.17}$$

where $\mathbf{v} = \mathbf{x}_t$ is the velocity vector of the fundamental flow, $\mathbf{V}$ is the velocity perturbation vector, P is the perturbation of the pressure, $\sigma > 0$ is the surface tension coefficient, $q^2 = R_1^{-2} + R_2^{-2}$, where R_1 and R_2 are the principal radii of curvature of normal cross sections of the surface Γ_t, $\partial p/\partial n \mid \Gamma_t$ is the derivative of p with respect to the outer normal to Γ_t, and $n = n(\xi)$ is the outer normal to Γ. The function $\rho(\xi, t)$ is given by the equation $\rho = |\nabla f|(|M^{*-1}\nabla f|)^{-1}$, where $f(\xi) = 0$ is the equation of Γ. It is assumed that the surface Γ belongs to the class C^3 and has a local parametrization of the form $\xi_i = \xi_i(\alpha, \beta)$ $(i = 1, 2, 3)$. Under the limiting condition (6.15) $\bar{\Delta}_\Gamma$ is the result of the transformation to Lagrangian coordinates of the Laplace–Beltrami operator Δ_{Γ_t} [19]:

$$\bar{\Delta}_\Gamma(t) = \frac{1}{\delta}\left\{\frac{\partial}{\partial\alpha}\left[\delta^{-1}\left(G\frac{\partial}{\partial\alpha} - F\frac{\partial}{\partial\beta}\right)\right] + \frac{\partial}{\partial\beta}\left[\delta^{-1}\left(E\frac{\partial}{\partial\beta} - F\frac{\partial}{\partial\alpha}\right)\right]\right\},$$

where $E = |M\xi_\alpha|^2$; $G = |M\xi_\beta|^2$; $F = M\xi_\alpha \cdot M\xi_\beta$, and $\delta = (EG - F^2)^{1/2}$. The operator $\bar{\Delta}_\Gamma(t)$ depends significantly on the time t.

The perturbation vector is recovered from the function $\mathbf{V}(\xi, t)$ by the equation

$$\mathbf{X} = M\int_0^t M^{-1}\mathbf{V}dt.$$

For irrotational body forces Eq. (6.14) has the integral

$$\mathbf{V} = M\frac{\partial}{\partial t}\left[M^{-1}W\int_0^t W^{-1}M^{*-1}\left(\mathbf{V}_0 - \int_0^\tau \nabla P d\mu\right)d\tau\right],$$

and by means of the replacement $\Phi_t = -P$ the problem reduces to a search for the single function $\Phi(\xi, t)$.

We note that the expression in the square brackets of Eq. (6.14) vanishes if the fundamental motion is irrotational (as follows from (6.4), this is possible only when $M \equiv W$). Furthermore, first-order differential operators are contained in Eqs. (6.13) and (6.14), and a second-order differential operator $\bar{\Delta}_\Gamma(t)$ is present in the boundary condition. Nevertheless, this problem reduces, similarly to the problem (6.1)-(6.3), to a nonlocal Cauchy problem in some Hilbert space. However, in contrast to the problem (6.1)-(6.3) it is correct when $\sigma > 0$ independently of the sign of $\partial p/\partial n|_\Gamma$. A priori estimates of the solution of the problem (6.13)-(6.17) of the energy-integral type are obtained in [27], and an existence and uniqueness theorem of a generalized solution of it is proven. This result confirms the role of surface tension as the regularizer of the problem of the motion of an ideal fluid with a free boundary.

In the particular case in which Γ does not depend on t and the fundamental motion is uniform rotation of the fluid as a rigid body, the problem (6.13)-(6.17) has been investigated in detail in [19].

The problem of small perturbations of the motion of a viscous capillary fluid has been studied in detail only in the case in which the fundamental motion is rest or uniform rotation [19]. Perturbations of arbitrary fundamental motion when $\sigma = 0$ and $\nu > 0$ have been discussed in [6].

A different formulation of the problem of perturbations of motion with a free boundary is possible: The region in which the mapping (1.7) is defined varies in the case of a constant initial velocity field. This problem was investigated in [13].

7. Stability of the Motion. Let some solution of the problem (1.9)-(1.11) defined for all $t \geq 0$ be known. Then it is possible to pose the question of the stability of this solution with respect to a change of the initial data. If the perturbations caused by this change are small, the problem of the stability can be discussed within the framework of the linear theory.

One should note that up to the present time there are no results of any kind on the solvability of the problem (1.9)-(1.11) on an infinite time interval. Therefore, the problem of the justification of the linear approximation in the theory of the stability of motion with a free boundary is now very far from solution.

If the fundamental solution is not time-independent, then the coefficients of Eqs. (6.13)-(6.17) depend on the time, which makes obtaining the sufficient conditions of stability from the linear approximation extremely difficult in the general case. All the results obtained up to the present time on the stability of nonsteady motions of a finite fluid mass are associated with the consideration of specific examples with a simple geometry of the free surface. In these examples the fluid is assumed to be ideal, and its motion is assumed to be irrotational (the exception is [28], in which the stability of a rotating ring of an ideal fluid is investigated).

First, let us discuss the perturbations of the irrotational motion of an ideal fluid with $\sigma=0$. In this case the problem reduces to a search for a function $\Psi(\xi, t)$ which satisfied (6.5) and (6.6). We will assume that the elements of the matrix M and the coefficient $a(\xi, t)$ are defined and are sufficiently smooth functions of ξ and t in the cylinder $\overline{Q}_T=\overline{\Omega}\times[0, T]$ for any $T>0$. The boundary Γ of the region Ω is also assumed to be sufficiently smooth. In addition, let the "hyperbolicity condition" (6.10) be satisfied for any $(\xi, t)\in\Gamma\times[0, T]$ with some $a_0(T)>0$ (it is assumed that $a_0\to 0$ as $T\to\infty$). This condition guarantees the correctness of the problem under consideration.

The estimate (6.11) is valid for the solutions of the problem (6.5)-(6.7). This estimate permits adopting the quantity $N(t)=\|\Psi_t\|^2_{L_2(\Gamma)}+\|\nabla\Psi\|^2_{L_2(\Omega)}$ as a measure of the stability and calling the fundamental solution stable if N(t) is bounded for all $t>0$ for any $\Phi_0\in W_2^1(\Omega)$ and unstable in the opposite case. Finally, such a definition of stability is not uniquely possible. It is proposed in [13] to characterize stability in terms of the behavior as $t\to\infty$ of the component normal to Γ_t of the perturbation vector of the free boundary $X(\xi, t)$, $\xi\in\Gamma$. In the general case it is given by the equation

$$R\equiv \mathbf{n}_{\Gamma_t}\cdot\mathbf{X}=\rho\int_0^t \mathbf{n}\cdot M^{-1}\nabla dt, \tag{7.1}$$

where $\rho=|\nabla f|(|M^{*-1}\nabla f|)^{-1}$, and $\mathbf{n}_{\Gamma_t}$ is the unit vector of the outer normal to the surface Γ at the point $\mathbf{x}(\xi, t)$.

If $R(\xi, t)\to\infty$ as $t\to\infty$ for some $\xi\in\Gamma$, then the local departure of the free surface from its unperturbed state increases without limit in the space (x). Thus the quantity $R(\xi, t)$ provides a very delicate characteristic of the stability of the motion, whereas the function N(t) is its crude integrated characteristic. It may happen that a certain solution is stable in the integrated sense, but the function R increases without limit at individual points of the boundary as $t\to\infty$. Such a situation actually arises in problems on the stability of motions with a linear velocity field described by Eqs. (3.1)-(3.6). In the particular case $g=0$, $b=0$, and $\mathbf{x}_0^0=0$,

$$\partial p/\partial n=-q(t)|N\xi|,\ q(t)=\mathrm{Sp}(A'A^{-1})^2/\mathrm{Sp}(A^{-1}A^{*-1}N).$$

The stability of this class of motion is investigated in more detail below. The integral identity ("energy integral")

$$\int_\Omega |U(\xi, t)|^2 d\Omega+\int_\Gamma\left|\frac{\partial p}{\partial n}\right|^{-1}|\Psi_t(\xi, t)|^2 d\Gamma=\int_\Omega|\nabla\varphi_0(\xi)|^2 d\Omega- \tag{7.2}$$

$$-2\int_0^t\int_\Omega U(\xi, t)\cdot A'A^{-1}U(\xi, t)\,d\Omega\,dt+\int_0^t\frac{q'}{q}\int_\Gamma\left|\frac{dp}{\partial n}\right|^{-1}|\Psi_t(\xi, t)|^2 d\Gamma\,dt$$

occurs, which is valid for any solution of the problem (6.5)-(6.7), where $U(\xi, t)=A^{*-1}(t)\nabla\Psi(\xi, t)$.

It is already possible to extract some information from the identity (7.2) about the behavior of $\|\Psi_t\|_{L_2(\Gamma)}$, e.g., for specific motions as $t\to\infty$ without solving the problem (6.5)-(6.7).

As an example, let us consider the two-dimensional problem of the stability of deformed ellipse. Here the fundamental solution is given by Eq. (3.1), where $x_0=0$ and $A=\mathrm{diag}(a_1, a_2)$. The functions $a_1(t)$ and $a_2(t)$ are defined by Eqs. (3.8), in which one sets $\omega=0$. Let us assume for definiteness that $b>0$; then the semimajor axis of the ellipse $a_1(t)$ is found from the equation

$$b\sqrt{2}\,t=\int_1^{a_1}(\tau^4+1)^{1/2}\frac{d\tau}{\tau^2}, \tag{7.3}$$

whence $a_1=b\sqrt{2t}+O(t^{-1})$ as $t\to\infty$. We obtain from the identity (7.2) the estimates

$$\int_\Gamma\Psi_t^2\,d\Gamma\leqslant\frac{4b^2ca_1^6}{(a_1^4+1)^2}\int_\Omega|\nabla\Phi_0|^2 d\Omega=O(t^{-2}),$$

$$\int_\Omega\Psi_{\xi_2}^2 d\Omega\leqslant\frac{a_1'}{a_1}\int_\Omega|\nabla\Phi_0|^2 d\Omega=O(t^{-1})$$

as $t\to\infty$, where Ω is a circle $|\xi|<c$, Γ is the circle $|\xi|=c$, and Φ_0 is a harmonic function in Ω.

More detailed information on the solution of the problem (6.5)-(6.7) is necessary to obtain the estimate $\|\Psi_{\xi_1}\|_{L_2(\Omega)}$. It turns out that the eigenfunctions of the operator K(t) do not depend on t [29] and the problem reduces to the Cauchy problem for a decomposing system of ordinary second-order differential equations. It has been shown in [29] that if $\Phi_0|_\Gamma\in L_2(\Gamma)$, the quantity $\|\Psi\|_{L_2(\Gamma)}$ is bounded for all $t>0$. If $\Phi_0|_\Gamma\in W_2^{1/2}(\Gamma)$, then Ψ_{tt}, Ψ_{ξ_1}, and Ψ_{ξ_2} belong to $L_2(\Gamma)$ for a fixed t, and their norms in $L_2(\Gamma)$ are bounded for all $t>0$. One should note that in the case in which the initial function Φ_0 is even with respect to ξ_1 and different from a constant, the estimate $\|\Psi\|_{L_2(\Gamma)} = O(t^{-1})$, $t \to \infty$ occurs. The solution which is even in ξ_1 describes motion with an impenetrable wall $\xi_2 = 0$.

Calculation of the normal component of the perturbation vector according to Eq. (7.1) for the case $\Phi_0|_\Gamma = \sin n\theta$, where $\theta = \arctan(\xi_2/\xi_1)$, gives

$$R = \frac{a_1^3 \sin n\theta}{(\cos^2\theta + a_1^4 \sin^2\theta)^{1/2}} [\gamma_n + O(t^{-4})] \tag{7.4}$$

as $t \to \infty$. Here γ_n is some constant, and $a_1(t)$ is a function defined by Eq. (7.3). It is evident from (7.4) that outside the zones $|\theta| < \varepsilon$, $|\pi - \theta| < \varepsilon$ ($\varepsilon > 0$ is specified) $R = O(t)$ as $t \to \infty$. Thus the free boundary is unstable in this case. Moreover, $\|\Psi\|_{L_2(\Gamma)} \to 0$ as $t \to \infty$.

If $\Phi_0|_\Gamma = \cos n\theta$, $n = 1, 2, \ldots$, then as $t \to \infty$

$$R = \frac{a_1 \cos n\theta}{(\cos^2\theta + a_1^4 \sin^2\theta)^{1/2}} [\delta_n + O(t^{-4})]$$

with constant δ_n. It follows from this that for this solution the instability of the free boundary is localized in the range of angles $|\theta| = O(t^{-1})$ and $|\pi - \theta| = O(t^{-1})$. This conclusion is applicable, in particular, to the problem of the stability of a deformed ellipse when an impenetrable wall $\xi_2 = 0$ is present. As $t \to \infty$ the fluid approaches the wall, and this stabilizes the free boundary.

Another example is the stability of irrotational axisymmetric motion with a linear velocity field. The fundamental solution is described in Sect. 3 and is interpreted as the motion of an ellipsoid of revolution. The mapping (3.1) has the form $x = \mathrm{diag}\{m, 1/\sqrt{m}, 1/\sqrt{m}\}\xi$ here. The function $m(t)$ is defined by Eq. (3.7), in which $\omega = 0$, and by the condition $m(0) = 1$. The region Ω is a sphere $|\xi| < c$, and Γ is its boundary. The identity (7.2) leads to the estimates

$$\|\Psi_{\xi_1}\|_{L_2(\Omega)} = O(1), \quad \|\Psi_t\|_{L_2(\Gamma)} = O(t^{-2})$$

as $t \to \infty$ for an oblate ellipsoid (which corresponds to $m \to 0$ as $t \to \infty$);

$$\|\Psi_{\xi_i}\|_{L_2(\Omega)} = O(1), \quad \|\Psi_t\|_{L_2(\Gamma)} = O(t^{-2}),$$

where $i = 1, 2$, for a prolate ellipsoid. Moreover, it is possible to point out initial data such that $R > kt$ as $t \to \infty$ with some constant $k > 0$ [30].

The results presented above indicate the stability for the linear approximation of the fundamental motions considered if one takes the norm in L_2 of the values of the potential Ψ or its derivatives as the measure of stability. However, if one assesses the stability from the departure of the free boundary from its unperturbed state, the indicated motions should be recognized as unstable.

Only irrotational perturbations have been considered above. If one conserves the irrotationality of the fundamental motion but widens the class of perturbations by removing the condition of irrotationality from them, then a solution which is stable with respect to irrotational perturbations may become unstable. Appropriate examples are given in [21]. The exact solutions discussed in Sec. 3, which describe the motions of a rotating ellipsoid and a rotating ellipse, indicate this behavior.

In addition to the cases discussed above, the stability of the following irrotational motions of an ideal fluid has been investigated in the linear approximation at present: the motion of a spherical layer [12] and the motion of a circular ring [13, 16].

Taking capillary forces into account in the problem of small perturbations leads, as has already been noted, to the correctly formulated problem (6.13)-(6.17). Concerning the effect of capillarity on stability, the number of specific problems considered here is very small. It is known that surface tension suppresses the growth of two-dimensional perturbations of the radial motion of a ring [16] and the motion of a rotating ring [28]. On the other hand, the introduction of surface tension results in the exponential growth as $t \to \infty$ of axisymmetric perturbations of the nonsteady motion of a fluid cylinder whose lateral surface is free and whose bases are solid impenetrable walls [31].

Problems of the stability of equilibrium states of a fluid volume are not considered in this paper. Sufficient conditions of stability and instability with respect to finite perturbations are obtained in this problem which are based on the results of [32], in which an analogue of the Lagrange stability theorem is established for the motion of a viscous capillary fluid. An exposition of this group of problems is contained in [19].

The problem of limiting modes of the motion of a fluid volume of finite mass as $t \to \infty$ is closely related to the stability problem. The examples discussed above of the motions of an ideal fluid and a viscous ring exhibit the diversity of the possibilities which arise here. This problem is far from solution in the general case. There are only the following partial results [9].

Let us assume that a classical solution of the problem (1.9)-(1.11) with $\mathbf{g}=0$ exists for all $t>0$, and that the mapping (1.7) specifies diffeomorphism of the regions Ω and Ω_t for any t. Let us denote the diameter of the region Ω_t as d(t), where

$$d(t) = \max\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in \Omega_t\}.$$

Let $\mathbf{v}_0(\xi) \neq 0$ and rot $\mathbf{v}_0 = 0$ for $\xi \in \Omega$. Then $d(t) \to \infty$ as $t \to \infty$.

In other words, in the case of the irrotational motion by inertia of a finite volume of ideal fluid with variable velocity and zero surface tension the diameter of the volume increases without limit as time goes by.

The proof of this result is based on the superharmonicity property of the pressure in the irrotational motion of the fluid and on the identity [33]

$$\frac{d^2}{dt^2}\int_{\Omega_t} |\mathbf{x}|^2 d\Omega_t = 2\int_{\Omega_t} |\mathbf{x}_t|^2 d\Omega_t - 4\sigma\int_{\Gamma_t} d\Gamma_t + 6\int_{\Omega_t} p\, d\Omega_t,$$

which is valid for arbitrary motion of an isolated volume of a viscous capillary fluid.

8. Boundary Layers. Let us assume that the solution of the problem (1.1)-(1.4) with $\nu>0$ is known. How does one find its asymptote as $\nu \to 0$? It is natural to expect that outside narrow layers near the free boundary the motion will be close to the motion of an ideal fluid. An abrupt change of the derivatives of the velocities occurs in the boundary layers which provides for vanishing of tangential stresses on the free boundary. A formal asymptote of the solution of this problem in the two-dimensional and axisymmetrical cases is given in [34], and the boundary layers in problems of the irrotational motion of an ellipsoid of revolution and an ellipse with $\sigma=0$ are investigated in [35, 36] (see Sec. 3).

The only example in which it has proven possible to justify the validity of an asymptotic expansion is the problem of a rotating ring. The asymptote of the solution is sought in the form

$$v_\theta \sim v_\theta^{(0)} + \sqrt{\nu}v_\theta^{(1)} + \sqrt{\nu}(\zeta_1^{(1)} + \zeta_2^{(1)}) + \nu v_\theta^{(2)} + \nu(\zeta_1^{(2)} + \zeta_2^{(2)}) + \ldots,$$

$$\chi \sim \chi^{(0)} + \sqrt{\nu}\chi^{(1)} + \nu\chi^{(2)} + \ldots,$$

$$r_i \sim r_i^{(0)} + \sqrt{\nu}r_i^{(1)} + \nu r_i^{(2)} + \ldots, \quad i = 1, 2.$$

The notation v_θ, χ, and r_i are introduced in Sec. 4. The functions $v_\theta^{(k)}$, $r_i^{(k)}$, and $\chi^{(k)}$ ($k=0, 1, 2, \ldots$) are found with the help of the first Lyusterni–Vishic iterative process. When $k=0$, we obtain the solution of the problem of the motion of a ring of ideal fluid. Functions of the boundary-layer type $\zeta_i^{(k)}$ ($i=1, 2, k=1, 2\ldots$) are determined as a result of the second iterative process. They compensate discrepancies on condition of the absence of tangential stress on the free boundaries of the ring. An estimate is given of the error of the asymptotic expansion as $\nu \to 0$ which is valid on any finite time interval if $\sigma=0$ and on any interval $0 \le t \le T < \infty$ on which $0 < \delta \le r_1^{(0)}(t)$ in the case $\sigma \neq 0$ [37].

The problem of constructing an asymptote of the solution of the problem (1.1)-(1.4) as $\nu \to 0$ in the general three-dimensional case remains open. Another unsolved problem is finding the asymptote of the solution of the problem of the motion of a finite fluid mass as $\sigma \to 0$.

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VORTICAL MOMENTUM OF BOUNDED IDEAL INCOMPRESSIBLE FLUID FLOWS

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1. The true momentum

$$\mathbf{I} \equiv \int \mathbf{v}\,dV$$

exists in three-dimensional homogeneous incompressible fluid flows filling the whole space and at rest at infinity only when the velocity field $\mathbf{v}(\mathbf{r}, t)$ satisfies the conditions [1]

$$r^3|\mathbf{v}(\mathbf{r}, t)| \to 0 \quad \text{as} \quad r \equiv |\mathbf{r}| \to \infty, \tag{1.1}$$

which excludes the important cases of flows possessing source and dipole asymptotics. If (1.1) is satisfied, then $\mathbf{I} = 0$.

Indeed

$$\int v_i\,dV = \int \frac{\partial}{\partial x_k}(x_i v_k)\,dV = \int x_i v_k\,dS_k. \tag{1.2}$$

The continuity equation and the rule of summation over repeated indices are used, and x_k are Cartesian coordinates.

The last integral in (1.2) is taken over an infinitely remote surface. It equals zero because of (1.1) so that $\mathbf{I} = 0$. Therefore, the true momentum either does not exist for the flows under consideration, or is zero.

For this reason, the so-called "vortical" momentum of the flow is introduced

$$\mathbf{P} \equiv \frac{1}{2}\int \mathbf{r} \times \boldsymbol{\omega}\,dV, \quad \boldsymbol{\omega} \equiv \operatorname{rot}\mathbf{v}. \tag{1.3}$$

This quantity was defined [2] only for fluid flows filling all space. It possesses the following properties:

a) It exists if $r^4|\boldsymbol{\omega}(\mathbf{r}, t)| \to 0$ as $r \to \infty$; this requirement is less constraining than (1.1) since it imposes a constraint on the behavior of the vortex field and not on the velocity at infinity;

b) it possesses the dimensionality of a momentum;

c) it is independent of the selection of the origin since $\int \boldsymbol{\omega}\,dV = 0$ in the case under consideration;

d) under the effect of external volume forces $\mathbf{f}(\mathbf{r}, t)$ it varies analogously to the physical momentum

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